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A General rational function : $f(x) = \frac{p(x)}{q(x)}$

where p, q are polynomial functions

$f(x)$ is proper if. $\deg p < \deg q$

otherwise it is improper.

A rational function can be written as

(*) — $f(x) = P(x) + \frac{g(x)}{q(x)}$ where $P(x)$, q, g are polynomials.

and $\frac{g}{q}$ is a proper rational

by " long division "

E.g,

$$\frac{2x^3 - 1}{x^2 + 1} = 2x + \frac{-2x - 1}{x^2 + 1}$$

quotient = $P(x)$

$$\begin{array}{r|l} & 2x \\ \hline x^2 \cdot + 1 & 2x^3 \quad\quad -1 \\ & 2x^3 + 2x \\ \hline & -2x - 1 \end{array}$$

remainder

E.g,

$$\frac{x^5 - 4x^4 + x^3 - 1}{x^2 + 2x + 5} = x^3 - 6x^2 + 8x + 14 + \frac{-68x - 71}{x^2 + 2x + 1}$$

$$\begin{array}{r|l} & x^3 - 6x^2 + 8x + 14 \\ \hline x^2 + 2x + 5 & x^5 - 4x^4 + x^3 \cdot \cdot -1 \\ & x^5 + 2x^4 + 5x^3 \\ \hline & -6x^4 - 4x^3 \quad\cdot\quad -1 \\ & -6x^4 - 12x^3 - 30x^2 \\ \hline & 8x^3 + 30x^2 \quad -1 \end{array}$$

$$\begin{array}{r} 8x^3+16x^2+40x \\ \hline 14x^2-40x-1 \\ 14x^2+28x+70 \\ \hline -68x-71 \end{array}$$ deg 1

- To integrate a general rational function, first use long division to write it in the form of (*).
  We know how to integrate polynomials,
  so next we need to know how to integrate general proper rational functions

Trick: "Partial Fraction Decomposition": break down a general proper rational function into certain simple types, then we discuss the integration of each type.

Fundamental Theorem of algebra:

Any polynomial can be factored into a product of linear factors and "irreducible" quadratic polynomials (i.e. quadratic polynomials which can not be factored as a product of linear factors)

E.g. $x^2+x=x(x+1)$ reducible

$x^2+2x+1=(x+1)^2$ reducible

x^2+2x+2 irreducible

ax^2+bx+c is irreducible if $b^2<4ac$. Otherwise it is reducible

$$(\dagger)\quad P(x) = C\,(x-a_1)^{n_1}(x-a_2)^{n_2}\cdots\; Q_1^{m_1} Q_2^{m_2}\cdots$$

where $n_1, n_2, \cdots m_1, m_2$ are integers

$a_1, a_2, \cdots$ all distinct.

$Q_1, Q_2, \cdots$ all distinct irreducible quadratic polynomials;

unique modulo a scalar factor (choice of c)

Partial Fraction Decomposition

$$P(x) = \frac{c_{11}}{x-a_1} + \frac{c_{12}}{(x-a_1)^2} + \cdots + \frac{c_{1n_1}}{(x-a_1)^{n_1}}$$

to integrate use substitution let $u = x-a_1$

$$+ \frac{c_{21}}{x-a_2} + \cdots + \frac{c_{2n_2}}{(x-a_1)^{n_2}}$$

$+ \cdots$ for all linear factors

$$+ \frac{b_{11}x+d_{11}}{Q_1} + \frac{b_{12}x+d_{12}}{Q_1^2} + \cdots + \frac{\text{linear function}}{Q_1^{m_1}}$$

let $u = Q_1$ combine substitution and Fact (**)

$$+ \frac{b_{21}x+d_{21}}{Q_2} + \cdots$$

for all quadratic factors

Eg,

$$\frac{2x+1}{x^2-2x-3}$$

First $x^2-2x-3 = (x+1)(x-3)$

Partial fraction decomposition:

$$\frac{2x+1}{x^2-2x-3} = \frac{a}{x+1} + \frac{b}{x-3} = \frac{a(x-3)}{(x+1)(x-3)} + \frac{b(x+1)}{(x+1)(x-3)}$$

$$= \frac{(a+b)x + b - 3a}{x^2-2x-3}$$

$$(a+b)x + b - 3a = 2x+1$$

plug in $x=0 \rightsquigarrow b-3a = 1$

$x=1 \rightsquigarrow a+b+b-3a = 3$

$b = 3a+1$

$$-2a + 2(3a+1) = 3$$

$$\rightsquigarrow 2(2a+1) = 3$$

$$\rightarrow 2a+1 = \frac{3}{2}$$

$$\rightarrow 2a = \frac{1}{2}$$

$$\rightarrow a = \frac{1}{4}, \qquad b = 1 + \frac{3}{4} = \frac{7}{4}$$

$$\int \frac{2x+1}{x^2-2x-3}\,dx = \int \frac{\frac{1}{4}}{\underbrace{x+1}_{=u}}\,dx + \int \frac{\frac{7}{4}}{\underbrace{x-3}_{=v}}\,dx$$

$$= \frac{1}{4}\int \frac{du}{u} + \frac{7}{4}\int \frac{dv}{v}$$

$du = dx$

$dv = dx$

$$= \frac{1}{4}\ln|x+1| + \frac{7}{4}\ln|x-3| + C$$

where C is an arbitrary constant ∎

E.g.,

$$\frac{5x+7}{(x-1)(x^2+x+2)}$$

irreducible

$$= \frac{c}{x-1} + \frac{ax+b}{x^2+x+2}$$

$$= \frac{c(x^2+x+2)}{(x-1)(x^2+x+2)} + \frac{(ax+b)(x-1)}{(x^2+x+2)(x-1)}$$

$$= \frac{(c+a)x^2 + (c-a+b)x + 2c - b}{(x-1)(x^2+x+2)}$$

$$5x+7 = (c+a)x^2 + (c-a+b)x + 2c - b$$

coefficients for x^2 term: $0 = c+a$

" x term $5 = c-a+b$

" constant " $7 = 2c-b$

3 equations, 3 unknowns a, b, c

solve for a, b, c.

$$\int \frac{5x+7}{(x-1)(x^2+x+2)}\,dx$$

$$= c\int \frac{1}{x-1}\,dx + \int \frac{ax+b}{(x^2+x+2)}\,dx$$

$$= c\ln|x-1| + \int \frac{ax+b}{(x^2+x+2)}\,dx$$

E.g.

$$\int \frac{2x+1}{x^2+x+2}\,dx \qquad x^2+x+2 = u.$$

$u = x^2+x+2$
$du = (2x+1)\,dx$

$$= \int \frac{du}{u}$$

$$= \ln|x^2+x+2| + C$$

where C is an arbitrary constant.

Fact **: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Fact: Any irreducible polynomial can be written as a "complete square" $c(x^2+a^2)$

E.g.

$$\int \frac{1}{x^2+x+2}\,dx$$

$$x^2+x+2 = \left(x+\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{7}}{2}\right)^2 = x^2+x+\tfrac{1}{4}+\tfrac{7}{4}$$

$$= \int \frac{1}{\underbrace{\left(x+\frac{1}{2}\right)}_{u}^2+\left(\underbrace{\frac{\sqrt{7}}{2}}_{a}\right)^2}\,dx$$

$u = x+\frac{1}{2}$

$du = dx$

$$= \int \frac{1}{u^2+\left(\frac{\sqrt{7}}{2}\right)^2}\,du$$

Fact

$$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2u}{\sqrt{7}}\right) + C$$

$$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right) + C \quad \square$$

E.g.

$$\int \frac{x^3-1}{x^2+x+2}\,dx$$

not proper! Can't use partial fraction decomposition directly! Use long division first.

$$\begin{array}{r l}
 & x-1 \\ \hline
x^2+x+2 \,) & x^3 \quad\quad -1 \\
 & x^3+x^2+2x \\ \hline
 & -x^2-2x-1 \\
 & -x^2-x-2 \\ \hline
 & -x+1
\end{array}$$

So $$\frac{x^3-1}{x^2+x+2} = x-1+\frac{-x+1}{x^2+x+2}$$

$$\int \frac{x^3-1}{x^2+x+2}\,dx = \int (x-1)\,dx + \int \frac{-x+1}{x^2+x+2}\,dx$$

$$= \frac{x^2}{2} - x + \int \frac{-\frac{1}{2}du+\frac{3}{2}dx}{\ldots}$$

$$= \frac{x^2}{2} - x - \frac{1}{2}\int \frac{du}{u} + \frac{3}{2}\int \frac{dx}{x^2+x+2}$$

$u = x^2+x+2$

$du = (2x+1)dx$

$-\frac{1}{2}du = (-x-\frac{1}{2})dx$

$$= \frac{x^2}{2} - x - \frac{1}{2}\ln|x^2+x+2| + \frac{3}{2}\left(\frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)\right) + C$$

where C is an arbitrary constant. $\square$

E.g. $\int \frac{2x+1}{x^3-2x^2}dx$

proper, can use partial fraction decomposition

$$\frac{2x+1}{x^3-2x^2}$$

$x^3-2x^2 = x^2(x-2)$

$$= \frac{C_1}{x} + \frac{C_2}{x^2} + \frac{C_3}{x-2}$$

$$= \frac{C_1 x\cdot(x-2)}{x\cdot x(x-2)} + \frac{C_2(x-2)}{x^2(x-2)} + \frac{C_3 x^2}{(x-2)x^2}$$

$$= \frac{(C_1+C_3)x^2 + (-2C_1+C_2)x - 2C_2}{x^2(x-2)}$$

$\rightsquigarrow \quad 2x+1 = (C_1+C_3)x^2 + (-2C_1+C_2)x - 2C_2.$

compare.: x^2 terms: $0 = C_1 + C_3$

x-terms: $2 = -2C_1 + C_2$.

constant terms: $1 = -2C_2$.

$\rightsquigarrow \quad C_2 = -\frac{1}{2}.$

$2C_1 = C_2 - 2 = -\frac{5}{2} \Rightarrow C_1 = -\frac{5}{4}$

$C_3 = -C_1 = \frac{5}{4}$

$$\int \frac{2x+1}{x^3-2x^2}\,dx$$

$$= -\frac{5}{4}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{dx}{x^2} + \frac{5}{4}\int \frac{dx}{x-2}$$

$$= -\frac{5}{4}\ln|x| - \frac{1}{2}\left(-\frac{1}{x}\right) + \frac{5}{4}\ln|x-2| + C$$

$\square$.